Problem 1 (35 pts)

We introduced the Motzkin polynomial in the lecture on Oct. 2nd, which is defined as:

$$p(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1\,. \tag{1}$$

The result you will prove is as follows:

Theorem 1. *$p(x, y) \geq 0$ for every $x, y \in \mathbb{R}$, while it cannot be expressed as a sum of squares of polynomials.*

1. **(5 pts)** Show that $p(x, y)$ is always non-negative.

To argue no SoS expression, we need an extra definition:

Definition 2 (Newton polytope)**.** For an n-variate polynomial $f(\mathbf{x}) = \sum_i c_i \mathbf{x}^{a_i}$, where $a_i \in \mathbb{N}^n$ and $c_i \in \mathbb{R}_{\neq 0}$, we assign the point a_i to the i-th term $c_i\mathbf{x}^{a_i}$. Then, the Newton polytope of f is the convex hull of these points in $\mathbb{R}^n$:

$$N(f) := \left\{ \sum_i \alpha_i a_i :\ \forall i\ \alpha_i \geq 0 \text{ and } \sum_i \alpha_i = 1 \right\}. \tag{2}$$

2. **(5 pts)** Prove that for two monomial terms a and b with assigned points p and q, the assigned point of ab is $p + q$.

3. **(5 pts)** For any SoS polynomial $f = \sum_i g_i^2$, show that $N(f) \subseteq 2X$, where X is the convex hull of all the points assigned to some terms in some g_i.
   $2X := \{2a : a \in X\} \subset \mathbb{R}^n$.

4. **(5 pts)** Prove that for two monomial terms a and b, if $ab = c^2$, then the assigned points p, q, r to a, b, c satisfy $p + q = 2r$.

5. **(5 pts)** For any SoS polynomial $f = \sum_i g_i^2$, show that $N(f) = 2X$, where X is the convex hull of all the points assigned to some terms in some g_i.

6. **(5 pts)** Draw the Newton polytope of the Motzkin polynomial $p(x, y)$ in $\mathbb{R}^2$, and show that the only possibility of its SoS expression has the following form:

$$p(x, y) = \sum_i (\alpha_i x^2 y + \beta_i xy^2 + \gamma_i xy + \delta_i)^2\,. \tag{3}$$

7. **(5 pts)** Finish the proof that $p(x, y)$ is not a sum-of-squares of polynomials.

Problem 2 (15 pts)

Show sum-of-squares proofs for the following inequalities:

1. **(5 pts)** Cauchy-Schwarz inequality:

$$\vdash \langle a, b\rangle \leq \frac{\epsilon}{2}\|a\|^2 + \frac{1}{2\epsilon}\|b\|^2 \tag{4}$$

and

$$\vdash \langle a, b\rangle^2 \leq \|a\|^2 \cdot \|b\|^2 \tag{5}$$

2. **(5 pts)** Triangle inequality:

$$\vdash (a+b+c)^2 \leq \frac{10}{3}(a^2+b^2+c^2) \tag{6}$$

3. **(5 pts)** Hölder inequality:

$$\vdash \sum_{i=1}^{d} a_i b_i^3 \leq \|a\|_4 \|b\|_4^3 \tag{7}$$

Problem 3 (15 pts)

We have introduced the Bloch sphere representation of a qubit in the lecture on Oct. 7th. More specifically, a qubit can be represented by a 3-dimensional Bloch vector $(a, b, c) \in \mathbb{R}^3$:

$$\rho = \frac{I - a\cdot X - b \cdot Y - c \cdot Z}{2}, \tag{8}$$

where $X, Y, Z \in \mathbb{C}^{2\times 2}$ are Pauli matrices.

1. **(5 pts)** Find a condition on the Bloch vector (a, b, c) such that ρ is a pure state, i.e., $\rho = |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle \in \mathbb{C}^2$.

2. **(10 pts)** Show that the density matrix representation and the Bloch sphere representation are equivalent.
Hint: the direction from the Bloch sphere to the density matrix requires some condition on the Bloch vector.

3. **(5 pts)** Verify that for the Quantum Max-Cut's local term

$$h_{ij} = \frac{I - X_iX_j - Y_iY_j - Z_iZ_j}{4} \tag{9}$$

on any edge $ij \in E$, and for any product state $\rho = \rho_i \otimes \rho_j$,

$$\mathrm{tr}[h_{ij}\rho] = \frac{1 - aa' - bb' - cc'}{4}, \tag{10}$$

where (a, b, c) and (a', b', c') are the Bloch vectors of ρ_i and ρ_j, respectively.